Does a Low Volatility Portfolio Need a "Low Volatility Anomaly?"

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Academic literature has observed over the last few decades that portfolios of low volatility stocks tend to earn higher risk-adjusted returns than portfolios of high-volatility stocks.[1] More recently, this observation has garnered much more attention both in academia as well as in practice, with many investment managers launching low-volatility strategies which seem to outperform cap-weighted indices over time and, in particular, over the past 10 to 15 years. The most commonly heard explanation for this "low volatility anomaly" appears to be that low risk stocks tend to outperform high risk stocks, or that low Beta stocks tend to outperform high Beta stocks over time.[2] Based on this explanation, many live strategies trying to exploit this "anomaly" construct portfolios by overweighting low volatility stocks or low Beta stocks and underweighting high volatility stocks or high beta stocks.

The goal of this paper is to provide an alternative explanation for the "low volatility anomaly." We will show that, by using only estimates of volatilities and correlations, an active manager can not only reduce portfolio volatility but also target varying portfolio long-run returns, even if all the stocks have the same long-run returns.

A Brief Review of Arithmetic and Geometric Returns

Consider the following two stocks in Figure 1 and Figure 2:

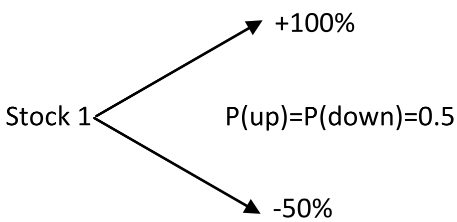


Figure 1

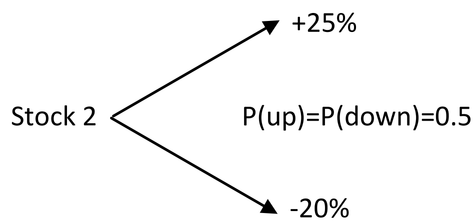


Figure 2

Stock 1 can move up 100% or down 50% in any given period with equal probability. Thus, it has an expected periodic arithmetic return of 25%. Stock 2 can move up 25% or down 20% in any given period with equal probability. Thus, it has an expected periodic arithmetic return of 2.5%. Moreover, a portfolio that is invested 80% in Stock 1 and 20% in Stock 2 will have an expected periodic arithmetic return of $0.8 \cdot 25\% + 0.2 \cdot 2.5\% = 20.5\%$.

However, over the long term, if Stock 1 doubles and halves with equal probability it has a zero percent expected long-run geometric return. Similarly if Stock 2 gains 25% and loses 20% with equal probability

[1] See for example Black, Jensen, and Scholes (1972) and Haugen and Baker (1991).
[2] See for example Baker, Bradley and Wurgler (2010) and Frazzini and Pedersen (2011).

it has a zero percent expected long-run geometric return $(1.25 \cdot 0.8 - 1 = 0)$. Therefore, simple intuition would suggest that a portfolio that is invested 80% in Stock 1 and 20% in Stock 2 would have a zero percent expected long-run geometric return. In fact, that same intuition suggests that any combination of those two stocks would yield zero percent over the long-run since $w \cdot 0\% + (1 - w) \cdot 0\% = 0\%$. We shall see below that when it comes to long-run geometric returns this simple intuition is misleading.

It should be fairly obvious from above that the expected arithmetic return of 25% for Stock 1 is not its long-run return and that the expected arithmetic return of 2.5% for Stock 2 is also not its long-run return. Stock 1 is more volatile than Stock 2. Over any single period Stock 1 has both greater upside and greater downside than Stock 2, but over the long-run both stocks are expected to remain flat.[3] It is important to keep in mind that, over the long-run, it is the geometric return that matters to investors because it reflects the true economic gain over time.

In the preceding paragraphs we have noted the importance on long-run geometric returns. We have also alluded to the possibility that combining two stocks with zero long-run geometric return does not necessarily have to yield a portfolio that is flat over the long-run. In the next section, we will show how this works out with this simple two stock example.

The Long-Run Geometric Return of a Portfolio

Consider the two stocks above where Stock 1 can go up 100% or down 50% with equal probability and Stock 2 can go up 25% and down 20% with equal probability. Over the long-run, both stocks have a zero compound return. We have hinted above that a portfolio invested 80% in Stock 1 and 20% in Stock 2 will not necessarily have a zero percent expected long-run return. Let us develop this idea a bit further since it will provide quite a bit of material for discussion.

Consider a portfolio that is invested 80% in Stock 1 and 20% in Stock 2 which rebalances to these weights after every period. Note that in this example there is no “low volatility anomaly” since the high volatility stock has the higher expected arithmetic return and, more importantly, both stocks have a zero expected long-run geometric return. Let us assume for discussion purposes that the two stocks are uncorrelated. Under this assumption, in any given period each of the following possibilities can occur with 25% probability:

Probability	0.25	0.25	0.25	0.25
Stock 1 Return	100%	100%	-50%	-50%
Stock 2 Return	25%	-20%	25%	-20%
80/20 Portfolio Return	85%	76%	-35%	-44%

Table 1

[3] For a lognormal distribution, the long-run return is equal to the arithmetic return minus one-half the variance ($\mu = a - 0.5\sigma^2$).

We discussed above that the expected arithmetic return for this portfolio is 0.8*25%+0.2*2.5%=20.5%. This can also be seen in Table 1 since ¼(85%+76%-35%-44%) =20.5%. The more important issue, however, is what the long-run geometric return of this portfolio is. Since each of the periodic returns can occur with equal probability, the long-run geometric return of the portfolio will be:

$$(1.85 \cdot 1.76 \cdot 0.65 \cdot 0.56)^{\frac{1}{4}} - 1 = 4.34\%.$$

This may seem somewhat surprising because each of the individual stocks has a zero expected long-run geometric return. Yet it appears that the long-run return of a portfolio of the two stocks is not 0.8*0%+0.2*0%, but rather 4.34%. It is also worth taking a note of the fact that the standard deviation of the periodic portfolio returns, 85%, 75%, -35% and -44%, is 60.2%.[4]

We have seen that the long-run expected return of the two-stock portfolio is greater than zero. Several important questions come to mind at this point. Will varying the portfolio weights affect the volatility of the portfolio? Will varying the portfolio weights of two stocks with zero expected long-run geometric return affect the long-run expected return of the portfolio? What are the variables which affect the volatility and long-run return of the portfolio?

Varying the Portfolio Weights

Let us explore what happens if we vary the portfolio weights to 50% invested in each stock. Now, in any given period each of the following possibilities can occur with 25% probability:

Probability	0.25	0.25	0.25	0.25
Stock 1 Return	100%	100%	-50%	-50%
Stock 2 Return	25%	-20%	25%	-20%
50/50 Portfolio Return	62.5%	40%	-12.5%	-35%

Table 2

Since each of the periodic returns can occur with equal probability, the expected long-run geometric return of the portfolio described in Table 2 will be:

$$(1.625 \cdot 1.4 \cdot 0.875 \cdot 0.65)^{\frac{1}{4}} - 1 = 6.65\%.$$

In addition, the standard deviation of the periodic portfolio returns of this portfolio, 62.5%, 40%, -12.5% and -35%, is 39.2%.

We should point out once again that in this example there is no "low volatility anomaly" since the high volatility stock has the higher expected arithmetic return and both stocks have a zero expected long-run geometric return. Yet, by increasing the weight on Stock 2, the less volatile stock, we were able to achieve both a higher long-run return and a lower volatility for the two stock portfolio. "Anomaly" or

[4] It is usually more appropriate to look at the volatility of the log returns but, for discussion purposes, we will be exploring the standard deviation of arithmetic returns.

not, this is a desirable result; can we do even better by increasing the weight on the less volatile stock even more?

Let us explore what happens if we now vary the portfolio weights to 20% in Stock 1 and 80% in Stock 2. Now, in any given period each of the following possibilities can occur with equal probability:

Probability	0.25	0.25	0.25	0.25
Stock 1 Return	100%	100%	-50%	-50%
Stock 2 Return	25%	-20%	25%	-20%
20/80 Portfolio Return	40%	4%	10%	-26%

Table 3

Since each of the periodic returns can occur with equal probability, the expected long-run geometric return of the portfolio in Table 3 will be:

$$(1.4 \cdot 1.04 \cdot 1.1 \cdot 0.74)^{\frac{1}{4}} - 1 = 4.34\%.$$

In addition, the standard deviation of the periodic portfolio returns, 40%, 4%, 10% and -26%, is 23.4%.

Alas, increasing the weight on the less volatile stock even further now reduced the expected long-run geometric return of the portfolio from 6.65% in the 50/50 case back to 4.34%. This was to be expected at some point since we know that a portfolio invested 100% in Stock 2 will have a zero expected long-run return.

Before we generalize this, it is also useful to explore what happens in a long/short portfolio. Suppose that we were to borrow 20% of our investment by selling short Stock 1 and buying more of Stock 2 or, in other words, the weight of stock 1 is -20% and the weight of Stock 2 is 120%.

Now, in any given period each of the following possibilities can occur with equal probability:

Probability	0.25	0.25	0.25	0.25
Stock 1 Return	100%	100%	-50%	-50%
Stock 2 Return	25%	-20%	25%	-20%
-20/120 Portfolio Return	10%	-44%	40%	-14%

Table 4

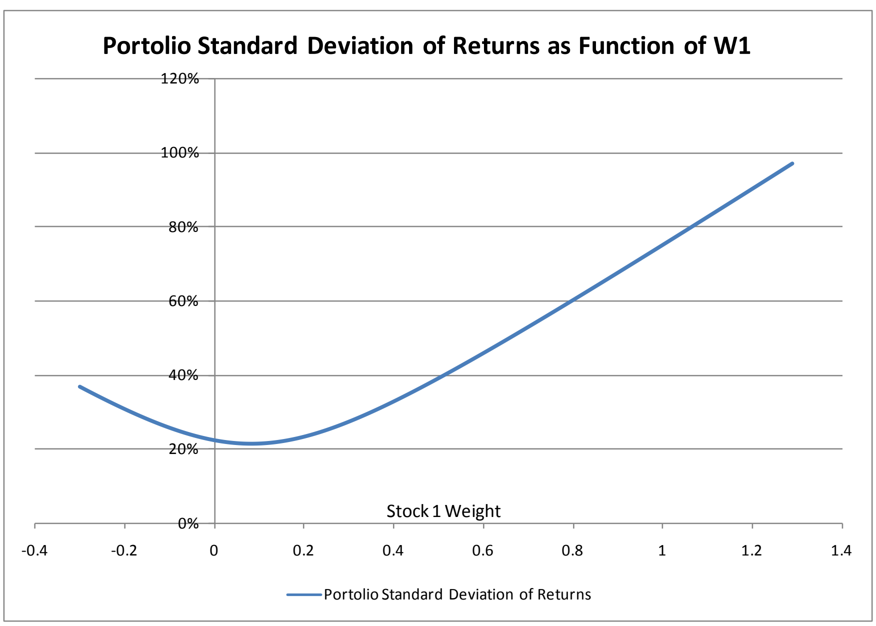
Since each of the periodic returns can occur with equal probability, the expected long-run geometric return of the portfolio in Table 4 will be:

$$(1.1 \cdot 0.56 \cdot 1.4 \cdot 0.86)^{\frac{1}{4}} - 1 = -7.20\%.$$

In this case, the portfolio will tend to lose money in the long run since it has a 25% chance of losing 44% in any given period and such an event requires almost an 80% return to get back to break even. The conclusion from this final exercise is that loading too much on low volatility stocks in a long/short portfolio does not immunize the portfolio against blowing up.

It is useful to explore how the expected long-run geometric return of the portfolio and the standard deviation of periodic returns vary as a function of the stock weights. In particular, since we have only two stocks, we can vary the weight on Stock 1, which would also determine the weight on Stock 2, and repeat the calculations above for the expected long-run return and standard deviation of portfolio returns.

In Figure 3, we can see that the volatility of the portfolio returns can be minimized at a weight of approximately 8% i.e., one would need to invest approximately 92% of the capital in Stock 2 (the less volatile stock) and the remaining 8% of the capital in Stock 1. This is a result of the well-known diversification effect of Modern Portfolio Theory. What may be more interesting are the results in the chart in Figure 4 which show that the expected long-run return is negative if the portfolio is not long-only, zero when we invest only in one of the stocks and positive for a long-only portfolio invested in both stocks. This is a key result which applies only to long-run returns and not to arithmetic returns: even though both stocks in the illustrative example above have a zero expected long-run geometric return, any long-only portfolio of the two stocks with non-zero weights will generate a portfolio that has a positive annualized return in the long-run. Moreover, the expected long-run return attains a maximum of 6.65% at 50/50 weights. Lastly, in Figure 5 we show how the risk-adjusted long-run return (Sharpe Ratio) varies as a function of the portfolio weights.[5] We can see that the risk-adjusted return is maximized when the weights are approximately 30% invested in Stock 1 and 70% invested in Stock 2.



[5] We compute risk-adjusted return for each weight combination as as long-run geometric return divided by the standard deviation of the periodic returns of the portfolio.

Figure 3

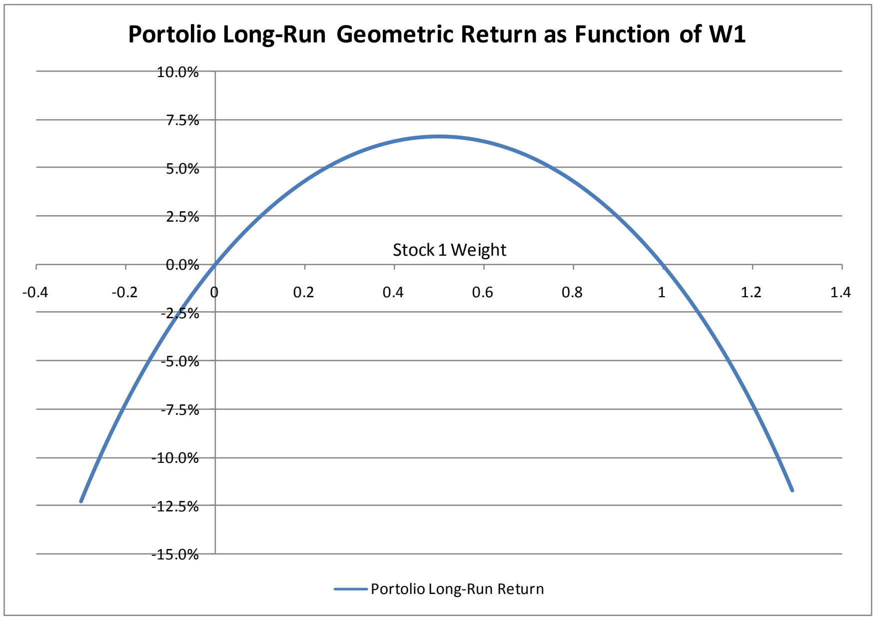


Figure 4

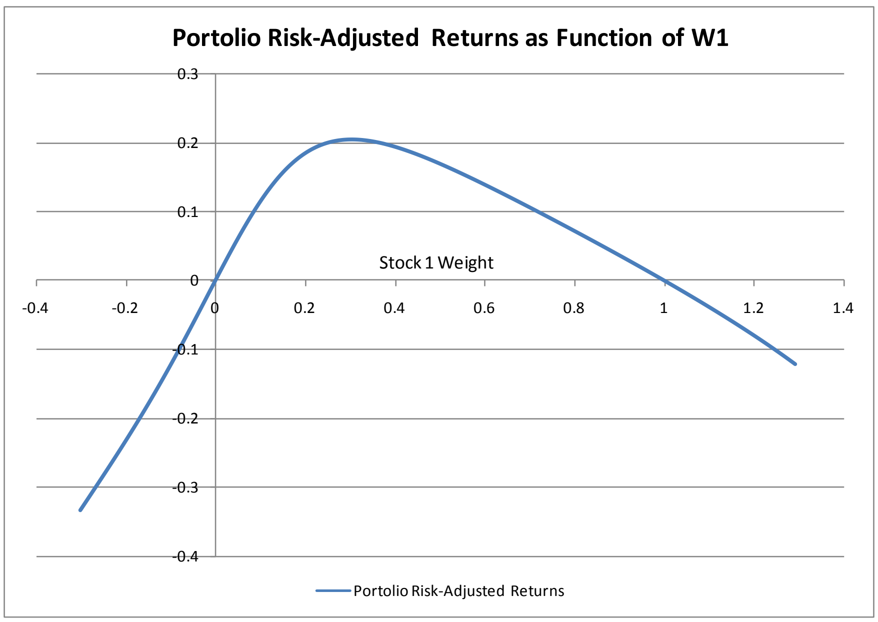


Figure 5

Key Takeaways

There are several important takeaways from the illustrative example above:

- When dealing with stock or portfolio returns, it is the long-run returns which end up affecting the bottom line of investors.

- Diversification is alive and well; it is possible to control the portfolio volatility by varying portfolio weights based on the variances and covariances of stocks.
- More importantly, it is possible to also target varying portfolio long-run geometric returns or long run excess returns even if all stocks have the same long-run geometric return (zero in the example above).

The result in the last bullet is a key result of Stochastic Portfolio Theory (Fernholz (1982) and Fernholz (2002)) and its applications are vast. Consider simple periodic (arithmetic) returns – if two stocks have an expected arithmetic return of 10% over the next year, any portfolio of the two stocks will have a 10% arithmetic return over the next year. Conversely, if the two stocks have an expected 10% long-run geometric return it is possible to construct a portfolio of the two stocks that has a long-run geometric return that exceeds 10%. And, what may be even more surprising is that the level of that excess long-run return depends only on stocks' volatilities and correlations.[6]

The intuition above is key to understanding how a manager like INTECH, which utilizes the mathematics of Stochastic Portfolio Theory, manages its active strategies and has managed to outperform cap weighted indices in several live strategies with excess returns that approximate targets set two decades ago. As for the need to have a low volatility anomaly to construct low volatility portfolios consider, for example, INTECH's Low Volatility strategies which target a return slightly above the market's return gross of fees (or market like returns net of fees) and attempt to minimize the portfolio volatility for that level of excess return. The example above demonstrates how it is possible to actually target a certain level of return or excess return and also control the portfolio volatility using only estimates of volatilities and correlations.[7] The combination of controlling the portfolio return and volatility also allows the control of the risk-adjusted return of the portfolio, or Sharpe Ratio.[8] Moreover, the example above also shows that it is possible to do this without relying on any behavioral "anomaly" since both stocks in the example had the same long-run geometric return (and the less volatile stock had a lower arithmetic return). To the extent that a "low volatility" anomaly does exist, INTECH will also benefit from that tailwind due to the characteristics of its low volatility portfolios but if the "low volatility anomaly" is transient, or gets arbitraged away, or doesn't even exist at all, INTECH believes that it should still be able to successfully manage low volatility portfolios using the concepts of Stochastic Portfolio Theory which do not rely on any form of behavioral anomalies.

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[6] Stochastic Portfolio Theory shows that, in the example above, the portfolio return will be approximately equal to the "excess growth rate" which is one-half times the weighted stock variance minus one-half of the portfolio variance.

[7] This requires some general assumptions about the long-term behavior and stability of the stock market.

[8] This also allows the control of the Information Ratio in strategies focused on relative risk.

Appendix

In this section we will illustrate some of the mathematical concepts within Stochastic Portfolio Theory and attempt to provide some additional intuition about those concepts.

If future stock prices are lognormally distributed, then the relationship between arithmetic and long-run geometric (compound) returns is as follows:

$$\begin{pmatrix} Compound \\ Return \end{pmatrix} = \begin{pmatrix} Expected \\ Arithmetic\ Return \end{pmatrix} - \frac{1}{2}\begin{pmatrix} Variance\ of \\ Returns \end{pmatrix}$$

This relationship will be exact in continuous time and approximate in discrete time. We should also point out that "variance of returns" is the variance of log returns. For example, consider Stock 1 from Figure 1. It can move either up 100% or down 50% with equal probability and therefore has an expected arithmetic return of 25%. In logarithmic terms, those returns are ln(2) and ln(0.5) which are approximately 0.69 and -0.69 respectively. Thus, the expected log return is zero and the standard deviation is approximately 69%. We have seen that the long run geometric return is zero and using the formula above we get that the Stock's compound return is $25\%-0.5*(69\%)^2 = 1\%$, which is pretty close to zero.

If the relationship above holds for stocks and portfolios, we can write that:

$$\begin{pmatrix} Weighted\ Average \\ Stock\ Compound\ Return \end{pmatrix} = \begin{pmatrix} Weighted\ Average \\ Stock\ Arithmetic\ Return \end{pmatrix} - \frac{1}{2}\begin{pmatrix} Weighted\ Average \\ Stock\ Variance \end{pmatrix}$$

and

$$\begin{pmatrix} Portfolio \\ Compound\ Return \end{pmatrix} = \begin{pmatrix} Expected\ Portfolio \\ Arithmetic\ Return \end{pmatrix} - \frac{1}{2}\begin{pmatrix} Portfolio \\ Variance \end{pmatrix}$$

Note that the expected portfolio arithmetic return is equal to the weighted average stock arithmetic return so we can rewrite the first equation as follows:

$$\begin{pmatrix} Weighted\ Average \\ Stock\ Arithmetic\ Return \end{pmatrix} = \begin{pmatrix} Expected\ Portfolio \\ Arithmetic\ Return \end{pmatrix} = \begin{pmatrix} Weighted\ Average \\ Stock\ Compound\ Return \end{pmatrix} + \frac{1}{2}\begin{pmatrix} Weighted\ Average \\ Stock\ Variance \end{pmatrix}$$

And substitute in the second equation to get the following:

$$\begin{pmatrix} Portfolio \\ Compound\ Return \end{pmatrix} = \begin{pmatrix} Weighted\ Average \\ Stock\ Compound\ Return \end{pmatrix} + \frac{1}{2}\left(\begin{pmatrix} Weighted\ Average \\ Stock\ Variance \end{pmatrix} - \begin{pmatrix} Portfolio \\ Variance \end{pmatrix}\right)$$

Or, alternatively,

$$\begin{pmatrix} Portfolio \\ Compound\ Return \end{pmatrix} = \begin{pmatrix} Weighted\ Average \\ Stock\ Compound\ Return \end{pmatrix} + (Excess\ Growth\ Rate)$$

Where the excess growth rate is $\frac{1}{2}\left(\begin{pmatrix} Weighted\ Average \\ Stock\ Variance \end{pmatrix} - \begin{pmatrix} Portfolio \\ Variance \end{pmatrix}\right)$

From the above equation, it can be noted that the excess growth rate depends only on volatilities and correlations. Stochastic portfolio theory also shows that the excess growth rate is positive for a long-only portfolio. Hence, by varying portfolio weights and, rebalancing to those weights, it is possible to achieve a long-run return above the weighted average stock compound return. Moreover, if varying the portfolio weights does not materially impact the weighted average stock compound return in the long-run, it is possible for an active manager to both target certain levels of excess return above a cap weighted index and control the portfolio volatility using only estimates of volatilities and correlations.

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